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D-CONCURRENT VECTOR FIELDS IN A FINSLER SPACE OF FIVE-DIMENSIONS

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ABSTRACT

The purpose of the present paper is to define and study D-concurrent vector fields in a Finsler space of five-dimensions. In this paper, D-concurrent vector fields of first kind based on D-tensors of first kind in a Finsler space of Five-dimensions have been defined and studies. The expressions for h- and v-covariant differentiations of D-tensor of first kind have also been obtained. Besides this, the Q-concurrent vector field in a five-dimensional Finsler space based on ^1Q-tensor is defined in this paper. Furthermore, a curvature tensor $^1D_{ijkh}$ based on D-tensor is also defined, its expression obtained and some properties studied.

KEYWORDS: *D-Concurrence, Curvatures and Five-Dimensional Finsler Space*

INTRODUCTION

In (1950), Tachibana [12] was the first author, who defined and studied concurrent vector fields in an n-dimensional Finsler space. This study was further taken up in (1974) by Matsumoto and Eguchi [3]. In (2004) while studying the existence of concurrent vector fields in a Finsler space Rastogi and Dwivedi [5] found that the definition of concurrent vector fields given earlier does not hold good, which led them to modify the definition of concurrent vector fields in Finsler space F^n. Recently, Rastogi [6] has defined and studied three kind of D-tensors, in a Finsler space of five-dimensions. In (2019) and (2020) Rastogi [7, 8; 9, 10], defined several new concurrent vector fields including D-concurrent vector fields in a Finsler space of three and four dimensions.

Let F^5, be a Finsler space of five-dimensions equipped with a fundamental function L (x, y), orthonormal frame $e_{\alpha)I}$, (α = 1,2,3,4,5), metric tensor g_{ij} and angular metric tensor h_{ij} given by [1], [6]

$$g_{ij} = l_i\, l_j + m_i\, m_j + n_{(1)I}\, n_{(1)j} + n_{(2)I}\, n_{(2)j} + n_{(3)I}\, n_{(3)j} \tag{1}$$

and

$$h_{ij} = m_i\, m_j + n_{(1)I}\, n_{(1)j} + n_{(2)I}\, n_{(2)j} + n_{(3)I}\, n_{(3)j} \tag{2}$$

Where, l_i, m_i, $n_{(1)I}$, $n_{(2)I}$ and $n_{(3)I}$ are five orthonormal vectors, alternatively expressed as $e_{1)I}$, $e_{2)I}$, $e_{3)I}$, $e_{4)I}$ and $e_{5)i}$.

The h-covariant derivative $e_{\alpha)}{}^{i}{}_{/j}$ of the vector $e_{\alpha)}$ is given as [4], [6]

$$e_{\alpha)}{}^{i}{}_{/j} = H_{\alpha)\beta\gamma}\, e_{\beta)}{}^{I}\, e_{\gamma)j} \tag{3}$$

Where, $H_{\alpha)\beta\gamma}$ are the scalar components of the h-covariant derivative given by (1.2) and are called h-connection scalars and satisfy

$$H_{\alpha)\beta\gamma} = -\, H_{\beta)\alpha\gamma} = H_{\alpha)\alpha\gamma} = 0 \tag{4}$$

Furthermore, using the definition

$H_{2)3\beta}\, e_{\beta)}{}^{j} = h_j = h_{\beta)}\, e_{\beta)j}$, $H_{4)2\beta}\, e_{\beta)}{}^{j} = j_j = j_\beta\, e_{\beta)j}$, $H_{3)4\beta}\, e_{\beta)}{}^{j} = k_j = k_\beta\, e_{\beta)j}$,

$$H_{5)2\beta}\, e_{\beta)}{}^{j} = r_j = r_\beta\, e_{\beta)j},\ H_{5)3\beta}\, e_{\beta)}{}^{j} = s_j = s_\beta\, e_{\beta)j},\ H_{5)4\beta}\, e_{\beta)}{}^{j} = t_j = t_\beta\, e_{\beta)j} \quad (5)$$

We can obtain on simplification $e_{1)}{}^{i}{}_{/j} = l^{i}{}_{/j} = 0$,

$$e_{2)}{}^{i}{}_{/j} = m^{i}{}_{/j} = n_{(1)}{}^{I}\, h_j - n_{(2)}{}^{I}\, j_j - n_{(3)}{}^{I}\, r_j,\ e_{3)}{}^{i}{}_{/j} = n_{(1)}{}^{i}{}_{/j} = n_{(2)}{}^{I}\, k_j - m^{i}\, h_j - n_{(3)}{}^{I}\, s_j$$

$$e_{4)}{}^{i}{}_{/j} = n_{(2)}{}^{i}{}_{/j} = m^{i}\, j_j - n_{(1)}{}^{I}\, k_j - n_{(3)}{}^{I}\, t_j,\ e_{5)}{}^{i}{}_{/j} = n_{(3)}{}^{i}{}_{/j} = m^{i}\, r_j + n_{(1)}{}^{I}\, s_j + n_{(2)}{}^{I}\, t_j \quad (6)$$

The v-covariant derivative of these vectors belonging to Miron frame $e_{\alpha)}$ can be given as [7]

$$e_{\alpha)}{}^{i}{}_{//j} = L^{-1}\, V_{\alpha)\beta\gamma}\, e_{\beta)}{}^{I}\, e_{\gamma)j} \quad (7)$$

Let $V_{\alpha)\beta\gamma}$ be scalar components of the v-covariant derivative given by (7) then $V_{\alpha)\beta\gamma}$ are called v-connection scalars. These scalars satisfy

$$V_{\alpha)\beta\gamma} = -\, V_{\beta)\alpha\gamma},\ V_{1)\beta\gamma} = \delta_{\beta\gamma} - \delta_{1\beta}\, \delta_{1\gamma} \quad (8)$$

Using equation (1.6), we can write

$$V_{1)1\gamma} = V_{2)2\gamma} = V_{3)3\gamma} = V_{4)4\gamma} = V_{5)5\gamma} = 0, \quad (9)$$

$$V_{1)2\gamma} = \delta_{2\gamma},\ V_{1)3\gamma} = \delta_{3\gamma},\ V_{1)4\gamma} = \delta_{4\gamma},\ V_{1)5\gamma} = \delta_{5\gamma}, \quad (10)$$

$$V_{2)1\gamma} = -\, \delta_{2\gamma},\ V_{2)3\gamma} = Q_\gamma,\ V_{2)4\gamma} = R_\gamma,\ V_{2)5\gamma} = S_\gamma, \quad (11)$$

$$V_{3)1\gamma} = -\, \delta_{3\gamma},\ V_{3)2\gamma} = -\, Q_\gamma,\ V_{3)4\gamma} = U_\gamma,\ V_{3)5\gamma} = V_\gamma, \quad (12)$$

$$V_{4)1\gamma} = -\, \delta_{4\gamma},\ V_{4)2\gamma} = -\, R_\gamma,\ V_{4)3\gamma} = -\, U_\gamma,\ V_{4)5\gamma} = X_\gamma, \quad (13)$$

$$V_{5)1\gamma} = -\, \delta_{5\gamma},\ V_{5)2\gamma} = -\, S_\gamma,\ V_{5)3\gamma} = -\, V_\gamma,\ V_{5)4\gamma} = -\, X_\gamma, \quad (14)$$

Where, we have defined and assumed Q_γ, R_γ, S_γ, U_γ, V_γ, X_γ, as the v-connection vectors.

Using equation (7), we can obtain

$$L\, e_{1)}{}^{i}{}_{//j} = L\, l^{i}{}_{//j} = m^{i}\, m_j + n_{(1)}{}^{I}\, n_{(1)j} + n_{(2)}{}^{I}\, n_{(2)j} + n_{(3)}{}^{I}\, n_{(3)j} = h^{i}{}_{j} \quad (15)$$

$$L\, e_{2)}{}^{i}{}_{//j} = L\, m^{i}{}_{//j} = -l^{i}\, m_j + n_{(1)}{}^{I}\, Q_j + n_{(2)}{}^{I}\, R_j + n_{(3)}{}^{I}\, S_j \quad (16)$$

$$L\, e_{3)}{}^{i}{}_{//j} = L\, n_{(1)}{}^{i}{}_{//j} = -\, l^{i}\, n_{(1)j} - m^{i}\, Q_j + n_{(2)}{}^{I}\, U_j + n_{(3)}{}^{I}\, V_j \quad (17)$$

$$L\, e_{4)}{}^{i}{}_{//j} = L\, n_{(2)}{}^{i}{}_{//j} = -\, l^{i}\, n_{(2)j} - m^{i}\, R_j - n_{(1)}{}^{I}\, U_j + n_{(3)}{}^{I}\, X_j \quad (18)$$

$$L\, e_{5)}{}^{i}{}_{//j} = L\, n_{(3)}{}^{i}{}_{//j} = -\, l^{i}\, n_{(3)j} - m^{i}\, S_j - n_{(1)}{}^{I}\, V_j - n_{(2)}{}^{I}\, X_j \quad (19)$$

The tensor C_{ijk} in F^5, is given by Rastogi [6] as follows:

$$L\, C_{ijk} = C_{(1)}\, m_i\, m_j\, m_k + C_{(2)}\, n_{(1)I}\, n_{(1)j}\, n_{(1)k} + C_{(3)}\, n_{(2)I}\, n_{(2)j}\, n_{(2)k} + C_{(4)}\, n_{(3)I}\, n_{(3)j}\, n_{(3)k}$$

$$+ \sum_{(I,j,k)} [C_{(5)}\, m_i\, m_j\, n_{(1)k} + C_{(6)}\, m_i\, m_j\, n_{(2)k} + C_{(7)}\, m_i\, m_j\, n_{(3)k} + C_{(8)}\, n_{(1)I}\, n_{(1)j}\, m_k$$

$$+ C_{(9)}\, n_{(1)I}\, n_{(1)j}\, n_{(2)k} + C_{(10)}\, n_{(1)I}\, n_{(1)j}\, n_{(3)k} + C_{(11)}\, n_{(2)I}\, n_{(2)j}\, m_k + C_{(12)}\, n_{(2)I}\, n_{(2)j}\, n_{(1)k}$$

$$+ C_{(13)} n_{(2)I} n_{(2)j} n_{(3)k} + C_{(14)} n_{(3)I} n_{(3)j} m_k + C_{(15)} n_{(3)I} n_{(3)j} n_{(1)k} + C_{(16)} n_{(3)I} n_{(3)j} n_{(2)k}$$

$$+ C_{(17)} m_i(n_{(1)j} n_{(2)k} + n_{(1)k} n_{(2)j}) + C_{(18)} m_i(n_{(1)j} n_{(3)k} + n_{(1)k} n_{(3)j})$$

$$+ C_{(19)} m_i (n_{(2)j} n_{(3)k} + n_{(2)k} n_{(3)j}) + C_{(20)} n_{(1)i}(n_{(2)j} n_{(3)k} + n_{(2)k} n_{(3)j})] \quad (20)$$

D-Concurrent Vector Field of First Kind

In a five-dimensional Finsler space F^5, there exist D-tensors of three kinds. Let ${}^1D_{ijk}$ be representing the D-tensor of first kind, which is such that [6]

$${}^1D_{ijk}\, l^i = 0 \text{ and } {}^1D_{ijk}\, g^{jk} = {}^1D_i = {}^1D\, n_{(1)I} \quad (21)$$

Then this tensor in F^5, can be expressed as

$${}^1D_{ijk} = D_{(1)} m_i m_j m_k + D_{(2)} n_{(1)I} n_{(1)j} n_{(1)k} + D_{(3)} n_{(2)I} n_{(2)j} n_{(2)k} + D_{(4)} n_{(3)I} n_{(3)j} n_{(3)k}$$

$$+\sum_{(ijk)} [D_{(5)} \{m_i m_j n_{(1)k}\} + D_{(6)} \{m_i m_j n_{(2)k}\} + D_{(7)} \{m_i m_j n_{(3)k}\}$$

$$+ D_{(8)}\{n_{(1)I} n_{(1)j} m_k\} + D_{(9)} \{n_{(1)I} n_{(1)j} n_{(2)k}\} + D_{(10)} \{n_{(1)I} n_{(1)j} n_{(3)k}\}$$

$$+ D_{(11)} \{n_{(2)I} n_{(2)j} m_k\} + D_{(12)} \{n_{(2)I} n_{(2)j} n_{(1)k}\} + D_{(13)} \{n_{(2)I} n_{(2)j} n_{(3)k}\}$$

$$+ D_{(14)} \{n_{(3)I} n_{(3)j} m_k\} + D_{(15)} \{n_{(3)I} n_{(3)j} n_{(1)k}\} + D_{(16)} \{n_{(3)I} n_{(3)j} n_{(2)k}\}$$

$$+ D_{(17)} \{m_i(n_{(1)j} n_{(2)k} + n_{(1)k} n_{(2)j})\} + D_{(18)} \{m_i(n_{(2)j} n_{(3)k} + n_{(2)k} n_{(3)j})\}$$

$$+D_{(19)} \{m_i (n_{(3)j} n_{(1)k} + n_{(3)k} n_{(1)j})\} + D_{(20)} \{n_{(1)i}(n_{(2)j} n_{(3)k} + n_{(2)k} n_{(3)j})\}] \quad (22)$$

Multiplying equation (2.2) by g^{jk}, we obtain on simplification

$${}^1D_i = m_i(D_{(1)} + D_{(8)} + D_{(11)} + D_{(14)}) + n_{(1)I} (D_{(2)} + D_{(5)} + D_{(12)} + D_{(15)})$$

$$+ n_{(2)i}(D_{(3)} + D_{(6)} + D_{(9)} + D_{(16)}) + n_{(3)i}(D_{(4)} + D_{(7)} + D_{(10)} + D_{(13)}), \quad (23)$$

which by virtue of (2.1) leads to

$$D_{(1)} + D_{(8)} + D_{(11)} + D_{(14)} = 0,\ D_{(2)} + D_{(5)} + D_{(12)} + D_{(15)} = {}^1D,$$

$$D_{(3)} + D_{(6)} + D_{(9)} + D_{(16)} = 0,\ D_{(4)} + D_{(7)} + D_{(10)} + D_{(13)} = 0. \quad (24)$$

Let $X^i(x)$, be a vector field in F^5, which is expressible as

$$X^i(x) = \alpha\, l^i + \beta\, m^i + \gamma\, n_{(1)}{}^i + \Theta\, n_{(2)}{}^i + \varphi\, n_{(3)}{}^I, \quad (25)$$

where α, β, γ, Θ and φ are scalars.

Assuming $X^i{}_{/j} = - \delta^i{}_j$, from equation (3.5), by virtue of equations (1.5), we can obtain

$$\alpha_{/j} = - l_j,\ \beta_{/j} = \gamma\, h_j - \Theta\, j_j - \varphi\, r_j - m_j,\ \gamma_{/j} = \Theta\, k_j - \varphi\, s_j - \beta\, h_j - n_{(1)j},$$

$$\Theta_{/j} = \beta\, j_j - \gamma\, k_j - \varphi\, t_j - n_{(2)j},\ \varphi_{/j} = \beta\, r_j + \gamma\, s_j + \Theta\, t_j - n_{(3)j} \quad (26)$$

which leads to

$$\alpha_{/0} = - 1,\ \beta_{/0} = \gamma\, h_0 - \Theta\, j_0 - \varphi\, r_0,\ \gamma_{/0} = \Theta\, k_0 - \varphi\, r_0 - \beta\, h_0,$$

$$\Theta_{/0} = \beta\, j_0 - \gamma\, k_0 - \varphi\, t_0,\ \varphi_{/0} = \beta\, r_0 + \gamma\, s_0 + \Theta\, t_0 \tag{27}$$

Now we shall give

Def. 2.1.: A vector field $X^i(x)$, satisfying $X^i_{/j} = -\delta^i_j$, given by equation (2.5), shall be called a D-concurrent vector field of first kind in a Finsler space of five-dimensions F^5, if for a scalar λ, it also satisfies

$$X^i\, {}^1D_{ijk} = \lambda\, h_{jk} \tag{28}$$

Using equations (22), (26) a, b and (27), we get

$$\lambda\, h_{jk} = m_j\, m_k\, \{\beta\, D_{(1)} + \gamma\, D_{(5)} + \Theta\, D_{(6)} + \varphi\, D_{(7)}\} + n_{(1)j}\, n_{(1)k}\, \{\beta\, D_{(8)} + \gamma\, D_{(2)} + \Theta\, D_{(9)} + \varphi\, D_{(10)}\}$$

$$+ n_{(2)j}\, n_{(2)k}\, \{\beta\, D_{(11)} + \gamma\, D_{(12)} + \Theta\, D_{(3)} + \varphi\, D_{(13)}\} + n_{(3)j}\, n_{(3)k}\, \{\beta\, D_{(14)} + \gamma\, D_{(15)} + \Theta\, D_{(16)} + \varphi\, D_{(4)}\}$$

$$+ (m_j\, n_{(1)k} + m_k\, n_{(1)j})\{\beta\, D_{(5)} + \gamma\, D_{(8)} + \Theta\, D_{(17)} + \varphi\, D_{(19)}\}$$

$$+ (m_j\, n_{(2)k} + m_k\, n_{(2)j})\ \{\beta\, D_{(6)} + \gamma\, D_{(17)} + \Theta\, D_{(11)} + \varphi\, D_{(18)}\}$$

$$+ (m_j\, n_{(3)k} + m_k\, n_{(3)j})\ \{\beta\, D_{(7)} + \gamma\, D_{(19)} + \Theta\, D_{(14)} + \varphi\, D_{(18)}\}$$

$$+ (n_{(1)j}\, n_{(2)k} + n_{(1)k}\, n_{(2)j})\ \{\beta\, D_{(17)} + \gamma\, D_{(19)} + \Theta\, D_{(12)} + \varphi\, D_{(20)}\}$$

$$+ (n_{(2)j}\, n_{(3)k} + n_{(2)k}\, n_{(3)j})\ \{\beta\, D_{(18)} + \gamma\, D_{(20)} + \Theta\, D_{(13)} + \varphi\, D_{(16)}\}$$

$$+ (n_{(3)j}\, n_{(1)k} + n_{(3)k}\, n_{(1)j})\ \{\beta\, D_{(19)} + \gamma\, D_{(10)} + \Theta\, D_{(20)} + \varphi\, D_{(15)}\} \tag{29}$$

Multiplying equation (29) by g^{jk} and using equation (2.4), we get on simplification

$$\lambda = (1/4)\, \gamma\, {}^1D \tag{30}$$

which by virtue of equations (6) and (29) also leads to

$${}^1D\, (\gamma_{/r} - \Theta\, k_r + \beta\, h_r + \varphi\, S_r) + {}^1D_r = 0 \tag{31}$$

Hence:

Theorem 2.1.: If $X^i(x)$ is a D-concurrent vector field of first kind in a five-dimensional Finsler space F^5, the scalar λ, is given by equation (30) and vector ${}^1D_r$ satisfies equation (31)

Multiplying equation (29) by m^j, $n_{(1)}{}^j$, $n_{(2)}{}^j$ and $n_{(3)}{}^j$, respectively, we get

$$\lambda\, m_k = m_k\, \{\beta\, D_{(1)} + \gamma\, D_{(5)} + \Theta\, D_{(6)} + \varphi\, D_{(7)}\} + n_{(1)k}\{\beta\, D_{(5)} + \gamma\, D_{(8)} + \Theta\, D_{(17)} + \varphi\, D_{(19)}\}$$

$$+ n_{(2)k}\{\beta\, D_{(6)} + \gamma\, D_{(17)} + \Theta\, D_{(11)} + \varphi\, D_{(18)}\} + n_{(3)k}\{\beta\, D_{(7)} + \gamma\, D_{(19)} + \Theta\, D_{(14)} + \varphi\, D_{(18)}\}, \tag{32}$$

$$\lambda\, n_{(1)k} = m_k\{\beta\, D_{(5)} + \gamma\, D_{(8)} + \Theta\, D_{(17)} + \varphi\, D_{(19)}\} + n_{(1)k}\{\beta\, D_{(8)} + \gamma\, D_{(2)} + \Theta\, D_{(9)} + \varphi\, D_{(10)}\}$$

$$+ n_{(2)k}\{\beta\, D_{(17)} + \gamma\, D_{(19)} + \Theta\, D_{(12)} + \varphi\, D_{(20)}\} + n_{(3)k}\{\beta\, D_{(19)} + \gamma\, D_{(10)} + \Theta\, D_{(20)} + \varphi\, D_{(15)}\} \tag{33}$$

$$\lambda\, n_{(2)k} = m_k\, \{\beta\, D_{(6)} + \gamma\, D_{(17)} + \Theta\, D_{(11)} + \varphi\, D_{(18)}\} + n_{(1)k}\{\beta\, D_{(17)} + \gamma\, D_{(19)} + \Theta\, D_{(12)} + \varphi\, D_{(20)}\}$$

$$+ n_{(2)k}\{\beta\, D_{(11)} + \gamma\, D_{(12)} + \Theta\, D_{(3)} + \varphi\, D_{(13)}\} + n_{(3)k}\{\beta\, D_{(18)} + \gamma\, D_{(20)} + \Theta\, D_{(13)} + \varphi\, D_{(16)}\}, \tag{34}$$

$$\lambda\, n_{(3)k} = m_k\, \{\beta\, D_{(7)} + \gamma\, D_{(19)} + \Theta\, D_{(14)} + \varphi\, D_{(18)}\} + n_{(1)k}\{\beta\, D_{(19)} + \gamma\, D_{(10)} + \Theta\, D_{(20)} + \varphi\, D_{(15)}\}$$

$$+ n_{(2)k}\{\beta D_{(18)} + \gamma D_{(20)} + \Theta D_{(13)} + \varphi D_{(16)}\} + n_{(3)k}\{\beta D_{(14)} + \gamma D_{(15)} + \Theta D_{(16)} + \varphi D_{(4)}\} \quad (35)$$

From these equations we can get

$$\lambda = \beta D_{(1)} + \gamma D_{(5)} + \Theta D_{(6)} + \varphi D_{(7)} = \beta D_{(8)} + \gamma D_{(2)} + \Theta D_{(9)} + \varphi D_{(10)}$$

$$= \beta D_{(11)} + \gamma D_{(12)} + \Theta D_{(3)} + \varphi D_{(13)} = \beta D_{(14)} + \gamma D_{(15)} + \Theta D_{(16)} + \varphi D_{(4)} \quad (36)$$

and

$$\beta D_{(5)} + \gamma D_{(8)} + \Theta D_{(17)} + \varphi D_{(19)} = \beta D_{(6)} + \gamma D_{(17)} + \Theta D_{(11)} + \varphi D_{(18)} = 0,$$

$$\beta D_{(7)} + \gamma D_{(19)} + \Theta D_{(14)} + \varphi D_{(18)} = \beta D_{(17)} + \gamma D_{(19)} + \Theta D_{(12)} + \varphi D_{(20)} = 0,$$

$$\beta D_{(19)} + \gamma D_{(10)} + \Theta D_{(20)} + \varphi D_{(15)} = \beta D_{(18)} + \gamma D_{(20)} + \Theta D_{(13)} + \varphi D_{(16)} = 0 \quad (37)$$

From equations given in (2.11) b, we can obtain after eliminating scalars β, γ, Θ and φ and some tedious calculation

$$E\,(CF - A\,G) + H\,(D\,E - B\,F) + I\,(A\,B - C\,D) = 0 \quad (38)$$

where we have substituted

$$A = D_{(8)} D_{(18)} - D^2_{(19)},\ B = D_{(11)} D_{(20)} - D_{(12)} D_{(18)},\ C = D_{(17)} D_{(20)} - D_{(18)} D_{(19)},$$

$$D = D_{(17)} D_{(18)} - D_{(14)} D_{(19)},\ E = D_{(6)} D_{(20)} - D_{(17)} D_{(18)},\ F = D_{(5)} D_{(18)} - D_{(7)} D_{(19)},$$

$$G = D_{(12)} D_{(16)} - D_{(13)} D_{(20)},\ H = D_{(16)} D_{(19)} - D^2_{(20)},\ I = D_{(16)} D_{(17)} - D_{(18)} D_{(20)}. \quad (39)$$

Hence:

Theorem 2.2.: If $X^i(x)$ is a D-concurrent vector field of first kind in a Finsler space of five-dimensions F^5, it satisfies equation (38), where coefficients A, B, C, D, E, F, G, H, I are given in terms of coefficients of ${}^1D_{ijk}$ by equation (39).

Weakly D-Concurrent Vector Fields

Multiplying equation (29) by m^k, $n_{(1)}{}^k$, $n_{(2)}{}^k$ and $n_{(3)}{}^k$, respectively, we get

$${}^1D_{ijk}\, m^k = D_{(1)} m_i m_j + D_{(5)}(m_i n_{(1)j} + m_j n_{(1)i}) + D_{(6)}(m_i n_{(2)j} + m_j n_{(2)i}) + D_{(7)}(m_i n_{(3)j} + m_j n_{(3)i})$$

$$+ D_{(8)} n_{(1)I} n_{(1)j} + D_{(11)} n_{(2)I} n_{(2)j} + D_{(14)} n_{(3)I} n_{(3)j} + D_{(17)}(n_{(1)I} n_{(2)j} + n_{(1)j} n_{(2)i})$$

$$+ D_{(18)}(n_{(2)I} n_{(3)j} + n_{(2)j} n_{(3)i}) + D_{(19)}(n_{(1)I} n_{(3)j} + n_{(1)j} n_{(3)i}), \quad (40)$$

$${}^1D_{ijk}\, n_{(1)}{}^k = D_{(2)} n_{(1)I} n_{(1)j} + D_{(5)} m_i m_j + D_{(8)}(m_i n_{(1)j} + m_j n_{(1)i}) + D_{(9)}(n_{(1)I} n_{(2)j} + n_{(1)j} n_{(2)i})$$

$$+ D_{(10)}(n_{(1)I} n_{(3)j} + n_{(1)j} n_{(3)i}) + D_{(121)} n_{(2)I} n_{(2)j} + D_{(15)} n_{(3)I} n_{(3)j} + D_{(17)}(m_i n_{(2)j} + m_j n_{(2)i})$$

$$+ D_{(19)}(m_i n_{(3)j} + m_j n_{(3)i}) + D_{(20)}(n_{(2)I} n_{(3)j} + n_{(2)j} n_{(3)i}), \quad (41)$$

$${}^1D_{ijk}\, n_{(2)}{}^k = D_{(3)} n_{(2)I} n_{(2)j} + D_{(6)} m_i m_j + D_{(9)} n_{(1)I} n_{(1)j} + D_{(11)}(m_i n_{(2)j} + m_j n_{(2)i})$$

$$+ D_{(12)}(n_{(1)I\ (2)j} + n_{(1)j} n_{(2)i}) + D_{(13)}(n_{(2)I} n_{(3)j} + n_{(2)j} n_{(3)i}) + D_{(16)} n_{(3)I} n_{(3)j}$$

$$+ D_{(17)}(m_i n_{(1)j} + m_j n_{(1)i}) + D_{(18)}(m_i n_{(3)j} + m_j n_{(3)i}) + D_{(20)}(n_{(1)I} n_{(3)j} + n_{(1)j} n_{(3)i}) \quad (42)$$

and

$$^{1}D_{ijk}\, n_{(3)}{}^{k} = D_{(4)}\, n_{(3)I}\, n_{(3)j} + D_{(7)}\, m_i\, m_j + D_{(10)}\, n_{(1)I}\, n_{(1)j} + D_{(13)}\, n_{(2)I}\, n_{(2)j} + D_{(14)}(m_i\, n_{(3)j} + m_j\, n_{(3)i})$$

$$+ D_{(15)}(n_{(1)I}\, n_{(3)j} + n_{(1)j}\, n_{(3)i}) + D_{(16)}(n_{(2)I}\, n_{(3)j} + n_{(2)j}\, n_{(3)i}) + D_{(18)}(m_i\, n_{(2)j} + m_j\, n_{(2)i})$$

$$+ D_{(19)}(m_i\, n_{(1)j} + m_j\, n_{(1)i}) + D_{(20)}(n_{(1)I}\, n_{(2)j} + n_{(1)j}\, n_{(2)i}). \quad (43)$$

These equations further give

$$^{1}D_{ijk}\, m^j\, m^k = {}^{11}D_i = D_{(1)}\, m_i + D_{(5)}\, n_{(1)I} + D_{(6)}\, n_{(2)I} + D_{(7)}\, n_{(3)I}, \quad (44)$$

$$^{1}D_{ijk}\, n_{(1)}{}^{j}\, m^k = {}^{21}D_i = D_{(5)}\, m_i + D_{(8)}\, n_{(1)I} + D_{(17)}\, n_{(2)I} + D_{(19)}\, n_{(3)I} = {}^{12}D_i, \quad (45)$$

$$^{1}D_{ijk}\, n_{(2)}{}^{j}\, m^k = {}^{31}D_i = D_{(6)}\, m_i + D_{(11)}\, n_{(2)I} + D_{(17)}\, n_{(1)I} + D_{(18)}\, n_{(3)I} = {}^{13}D_i, \quad (46)$$

$$^{1}D_{ijk}\, n_{(3)}{}^{j}\, m^k = {}^{41}D_i = D_{(7)}\, m_i + D_{(14)}\, n_{(3)I} + D_{(18)}\, n_{(2)I} + D_{(19)}\, n_{(1)I} = {}^{14}D_i, \quad (47)$$

$$^{1}D_{ijk}\, n_{(1)}{}^{j}\, n_{(1)}{}^{k} = {}^{22}D_i = D_{(2)}\, n_{(1)I} + D_{(8)}\, m_i + D_{(9)}\, n_{(2)I} + D_{(10)}\, n_{(3)I}, \quad (48)$$

$$^{1}D_{ijk}\, n_{(2)}{}^{j}\, n_{(1)}{}^{k} = {}^{32}D_i = D_{(9)}\, n_{(1)I} + D_{(12)}\, n_{(2)I} + D_{(17)}\, m_i + D_{(20)}\, n_{(3)I} = {}^{23}D_i, \quad (49)$$

$$^{1}D_{ijk}\, n_{(3)}{}^{j}\, n_{(1)}{}^{k} = {}^{42}D_i = D_{(10)}\, n_{(1)I} + D_{(15)}\, n_{(3)I} + D_{(19)}\, m_i + D_{(20)}\, n_{(2)I} = {}^{24}D_i, \quad (50)$$

$$^{1}D_{ijk}\, n_{(2)}{}^{j}\, n_{(2)}{}^{k} = {}^{33}D_i = D_{(3)}\, n_{(2)I} + D_{(11)}\, m_i + D_{(12)}\, n_{(1)I} + D_{(13)}\, n_{(3)I}, \quad (51)$$

$$^{1}D_{ijk}\, n_{(3)}{}^{j}\, n_{(2)}{}^{k} = {}^{43}D_i = D_{(13)}\, n_{(2)I} + D_{(16)}\, n_{(3)I} + D_{(18)}\, m_i + D_{(20)}\, n_{(1)I} = {}^{34}D_i, \quad (52)$$

$$^{1}D_{ijk}\, n_{(3)}{}^{j}\, n_{(3)}{}^{k} = {}^{44}D_i = D_{(4)}\, n_{(3)I} + D_{(14)}\, m_i + D_{(15)}\, n_{(1)I} + D_{(16)}\, n_{(2)I} \quad (53)$$

Hence:

Theorem 3.1.: In a five-dimensional Finsler space F^5, tensor $^{1}D_{ijk}$ gives ten vectors out of which four vectors are unique and are given by equations (44), (45), (48) and (53)

Now similar to [7], we shall give following definitions:

Weakly D-Concurrent Vector Fields of First Kind: A vector field $X^i(x)$, in a five-dimensional Finsler space F^5, shall be called weakly D-concurrent vector field of first kind, if for $X^i{}_{/j} = - \delta^i{}_j$, and a scalar function $\mu_{(1)}(x, y)$, $^{11}D_i$ given by equation (44) satisfies

$$X^i\, {}^{11}D_i = \mu_{(1)}(x, y) \quad (54)$$

Weakly D-Concurrent Vector Fields of Second Kind. A vector field $X^i(x)$, in a five-dimensional Finsler space F^5, shall be called weakly D-concurrent vector field of second kind, if for $X^i{}_{/j} = - \delta^i{}_j$, and a scalar function $\mu_{(2)}(x, y)$, $^{22}D_i$ given by equation (48) satisfies

$$X^i\, {}^{22}D_i = \mu_{(2)}(x, y) \quad (55)$$

Weakly D-Concurrent Vector Fields of Third Kind: A vector field $X^i(x)$, in a five-dimensional Finsler space F^5, shall be called weakly D-concurrent vector field of third kind, if for $X^i{}_{/j} = - \delta^i{}_j$, and a scalar function $\mu_{(3)}(x, y)$, $^{33}D_i$ given by equation (51) satisfies

$$X^i\, {}^{33}D_i = \mu_{(3)}(x, y) \quad (56)$$

Weakly D-Concurrent Vector Fields of Fourth Kind: A vector field $X^i(x)$, in a five-dimensional Finsler space F^5, shall be called weakly D-concurrent vector field of fourth kind, if for $X^i_{/j} = - \delta^i_j$, and a scalar function $\mu_{(4)}(x, y)$, ${}^{44}D_i$ given by equation (53) satisfies

$$X^i\, {}^{44}D_i = \mu_{(4)}(x, y) \quad (57)$$

Equations (54), (55), (56) and (57) with the help of equations (25) can be expressed as

$$\mu_{(1)}(x, y) = \beta D_{(1)} + \gamma D_{(5)} + \Theta D_{(6)} + \varphi D_{(7)},\ \mu_{(2)}(x, y) = \beta D_{(8)} + \gamma D_{(2)} + \Theta D_{(9)} + \varphi D_{(10)}, \quad (58)$$

$$\mu_{(3)}(x, y) = \beta D_{(11)} + \gamma D_{(12)} + \Theta D_{(3)} + \varphi D_{(13)},\ \mu_{(4)}(x, y) = \beta D_{(14)} + \gamma D_{(15)} + \Theta D_{(16)} + \varphi D_{(4)}. \quad (59)$$

Hence:

Theorem 3.2.: In a five-dimensional Finsler space F^5, weakly D-concurrent vector fields of first, second, third and fourth kind have scalars $\mu_{(1)}(x, y)$, $\mu_{(2)}(x, y)$, $\mu_{(3)}(x, y)$ and $\mu_{(4)}(x, y)$ satisfying equations (58) and (59).

Taking h-covariant derivatives of equations (58) and (59) with the help of equation (26) a, we get

$$\mu_{(1)/j} = \beta(D_{(1)/j} - D_{(5)} h_j + D_{(6)} j_j + D_{(7)} r_j) + \gamma(D_{(5)/j} + D_{(1)} h_j - D_{(6)} k_j + D_{(7)} s_j)$$
$$+ \Theta(D_{(6)/j} - D_{(1)} j_j + D_{(5)} k_j + D_{(7)} t_j) + \varphi(D_{(7)/j} - D_{(1)} r_j - D_{(5)} s_j - D_{(6)} t_j) - {}^{11}D_j, \quad (60)$$

$$\mu_{(2)/j} = \beta(D_{(8)/j} - D_{(2)} h_j + D_{(9)} j_j + D_{(10)} r_j) + \gamma (D_{(2)/j} + D_{(8)} h_j - D_{(9)} k_j + D_{(10)} s_j)$$
$$+ \Theta(D_{(9)/j} - D_{(8)} j_j + D_{(2)} k_j + D_{(10)} t_j) + \varphi(D_{(10)/j} - D_{(8)} r_j - D_{(2)} s_j - D_{(9)} t_j) - {}^{22}D_j, \quad (61)$$

$$\mu_{(3)/j} = \beta(D_{(11)/j} - D_{(12)} h_j + D_{(3)} j_j + D_{(13)} r_j) + \gamma(D_{(12)/j} + D_{(11)} h_j - D_{(3)} k_j + D_{(13)} s_j)$$
$$+ \Theta(D_{(3)/j} - D_{(11)} j_j + D_{(12)} k_j + D_{(13)} t_j) + \varphi(D_{(13)/j} - D_{(11)} r_j - D_{(12)} s_j - D_{(3)} t_j) - {}^{33}D_j, \quad (62)$$

$$\mu_{(4)/j} = \beta(D_{(14)/j} - D_{(15)} h_j + D_{(16)} j_j + D_{(4)} r_j) + \gamma(D_{(15)/j} + D_{(14)} h_j - D_{(16)} k_j + D_{(4)} s_j)$$
$$+ \Theta(D_{(16)/j} - D_{(14)} j_j + D_{(15)} k_j + D_{(4)} t_j) + \varphi(D_{(4)/j} - D_{(14)} r_j - D_{(15)} s_j - D_{(16)} t_j) - {}^{44}D_j \quad (63)$$

Which lead to

$$\mu_{(1)/0} = \beta(D_{(1)/0} - D_{(5)} h_0 + D_{(6)} j_0 + D_{(7)} r_0) + \gamma(D_{(5)/0} + D_{(1)} h_0 - D_{(6)} k_0 + D_{(7)} s_0)$$
$$+ \Theta(D_{(6)/0} - D_{(1)} j_0 + D_{(5)} k_0 + D_{(7)} t_0) + \varphi(D_{(7)/0} - D_{(1)} r_0 - D_{(5)} s_0 - D_{(6)} t_0), \quad (64)$$

$$\mu_{(2)/0} = \beta(D_{(8)/0} - D_{(2)} h_0 + D_{(9)} j_0 + D_{(10)} r_0) + \gamma (D_{(2)/0} + D_{(8)} h_0 - D_{(9)} k_0 + D_{(10)} s_0)$$
$$+ \Theta(D_{(9)/0} - D_{(8)} j_0 + D_{(2)} k_0 + D_{(10)} t_0) + \varphi(D_{(10)/0} - D_{(8)} r_0 - D_{(2)} s_0 - D_{(9)} t_0), \quad (65)$$

$$\mu_{(3)/0} = \beta(D_{(11)/0} - D_{(12)} h_0 + D_{(3)} j_0 + D_{(13)} r_0) + \gamma(D_{(12)/0} + D_{(11)} h_0 - D_{(3)} k_0 + D_{(13)} s_0)$$
$$+ \Theta(D_{(3)/0} - D_{(11)} j_0 + D_{(12)} k_0 + D_{(13)} t_0) + \varphi(D_{(13)/0} - D_{(11)} r_0 - D_{(12)} s_0 - D_{(3)} t_0), \quad (66)$$

$$\mu_{(4)/0} = \beta(D_{(14)/0} - D_{(15)} h_0 + D_{(16)} j_0 + D_{(4)} r_0) + \gamma(D_{(15)/0} + D_{(14)} h_0 - D_{(16)} k_0 + D_{(4)} s_0)$$
$$+ \Theta(D_{(16)/0} - D_{(14)} j_0 + D_{(15)} k_0 + D_{(4)} t_0) + \varphi(D_{(4)/0} - D_{(14)} r_0 - D_{(15)} s_0 - D_{(16)} t_0) \quad (67)$$

Hence:

Theorem 3.3.: In a five-dimensional Finsler space F^5, weakly D-concurrent vector fields of first, second, third and fourth kind have scalars whose h-covariant derivatives satisfy equations (60) and (64).

Remarks:

- It can be observed that D-concurrent vector field of first kind in a five-dimensional Finsler space shall give weakly D-concurrent vector fields of first, second, third and fourth kind, but the converse is not true in general.
- Similar to h-covariant derivatives, we can also obtain v-covariant derivatives of scalars defined above.

TENSOR $^1D_{ijk/r}$ IN F^5

Taking h-covariant derivative of equation (22) and using equation (6), we can obtain [6]

$$^1D_{ijk/h} = A_{(1)h}\, m_i\, m_j\, m_k + A_{(2)h}\, n_{(1)i}\, n_{(1)j}\, n_{(1)k} + A_{(3)h}\, n_{(2)i}\, n_{(2)j}\, n_{(2)k} + A_{(4)h}\, n_{(3)I}\, n_{(3)j}\, n_{(3)k}$$

$$+ \sum_{(I,j,k)} [A_{(5)h} \{ m_i\, m_j\, n_{(1)k}\} + A_{(6)h} \{ m_i\, m_j\, n_{(2)k}\} + A_{(7)h} \{ m_i\, m_j\, n_{(3)k}\}$$

$$+ A_{(8)h} \{ n_{(1)i}\, n_{(1)jj}\, m_k\} + A_{(9)h} \{ n_{(1)i}\, n_{(1)j}\, n_{(2)k}\} + A_{(10)h} \{n_{(1)i}\, n_{(1)j}\, n_{(3)k}\}$$

$$+ A_{(11)h} \{n_{(2)i}\, n_{(2)j}\, m_k\} + A_{(12)h} \{n_{(2)i}\, n_{(2)j}\, n_{(1)k}\} + A_{(13)h} \{n_{(2)i}\, n_{(2)j}\, n_{(3)k}\}$$

$$+ A_{(14)h} \{n_{(3)i}\, n_{(3)j}\, m_k\} + A_{(15)h} \{n_{(3)i}\, n_{(3)j}\, n_{(1)k}\} + A_{(16)h} \{ n_{(3)i}\, n_{(3)j}\, n_{(2)k}\}$$

$$+ A_{(17)h} \{m_i(n_{(1)j}\, n_{(2)k} + n_{(1)k}\, n_{(2)j})\} + A_{(18)h}\{m_i(n_{(1)j}\, n_{(3)k} + n_{(1)k}\, n_{(3)j})\}$$

$$+ A_{(19)h} \{m_i(n_{(2)j}\, n_{(3)k} + n_{(2)k}\, n_{(3)j})\} + A_{(20)h} \{n_{(1)i}(n_{(2)j}\, n_{(3)k} + n_{(2)k}\, n_{(3)j})\} \quad (4.1)$$

Where we have used

$$A_{(1)j} = D_{(1)/j} + 3(D_{(6)}\, h_{(3)j} – D_{(5)}\, h_{(1)j} + D_{(7)}\, h_{(4)j}),\ A_{(2)j} = D_{(2)/j} + 3(D_{(8)}\, h_{(1)j} – D_{(9)}\, h_{(2)j} + D_{(10)}\, h_{(5)j})$$

$$A_{(3)j} = D_{(3)/j} + 3(D_{(12)}h_{(2)j} – D_{(11)}\, h_{(3)j} + D_{(13)}\, h_{(6)j}),\ A_{(4)j} = D_{(4)/j} – 3(D_{(14)}\, h_{(4)j} + D_{(15)}\, h_{(5)j} + D_{(16)}\, h_{(6)j})$$

$$A_{(5)j} = D_{(5)/j} + (D_{(1)} – 2D_{(8)})h_{(1)j} – D_{(6)}\, h_{(2)j} + D_{(7)}\, h_{(5)j} + 2\, D_{(17)}\, h_{(3)j} + 2D_{(18)}\, h_{(4)j}$$

$$A_{(6)j} = D_{(6)/j} – (D_{(1)} – 2\, D_{(11)})h_{(3)j} + D_{(5)}\, h_{(2)j} + D_{(7)}\, h_{(6)j} – 2\, D_{(17)}\, h_{(1)j} + 2\, D_{(19)}\, h_{(4)}$$

$$A_{(7)j} = D_{(7)/j} -(D_{(1)} – 2\, D_{(14)})h_{(4)j} – D_{(5)}\, h_{(5)j} – D_{(6)}\, h_{(6)j} – 2\, D_{(18)}\, h_{(1)j} + 2\, D_{(19)}\, h_{(3)}$$

$$A_{(8)j} = D_{(8)/j} – (D_{(2)} – 2\, D_{(5)})h_{(1)j} + D_{(9)}\, h_{(3)j} + D_{(10)}\, h_{(4)j} – 2\, D_{(17)}\, h_{(2)j} + 2\, D_{(18)}\, h_{(5)j}$$

$$A_{(9)j} = D_{(9)/j} + (D_{(2)} – 2D_{(12)})\, h_{(2)j} – D_{(8)}\, h_{(3)j} + D_{(10)}\, h_{(6)j} + 2\, D_{(17)}\, h_{(1)j} + 2\, D_{(20)}\, h_{(5)j}$$

$$A_{(10)j} = D_{(10)/j} - (D_{(2)} – 2\, D_{(15)})h_{(5)j} – D_{(8)}\, h_{(4)j} - D_{(9)}\, h_{(6)j} + 2\, d_{(18)}\, h_{(1)j} – 2\, D_{(20)}\, h_{(2)j}$$

$$A_{(11)j} = D_{(11)/j} + (D_{(3)} - 2\, D_{(6)})h_{(3)j} - D_{(12)}\, h_{(1)j} + D_{(13)}\, h_{(4)j} + 2D_{(17)}\, h_{(2)j} + 2D_{(19)}\, h_{(6)j}$$

$$A_{(12)j} = D_{(12)/j} + D_{(11)}\, h_{(1)j} - (D_{(3)} - 2\, D_{(9)})h_{(2)j} + D_{(13)}\, h_{(5)j} – 2\, D_{(17)}\, h_{(3)j} + 2\, D_{(20)}\, h_{(6)j}$$

$$A_{(13)j} = D_{(13)/j} -(D_{(3)} – 2\, D_{(16)})\, h_{(6)j} – D_{(11)}\, h_{(4)j} – D_{(12)}\, h_{(5)j} – 2\, D_{(19)}\, h_{(3)j} + 2\, D_{(20)}\, h_{(2)j}$$

$$A_{(14)j} = D_{(14)/j} + (D_{(4)} – 2\, D_{(7)})h_{(4)j} – D_{(15)}\, h_{(1)j} + D_{(16)}\, h_{(3)j} – 2\, D_{(18)}\, h_{(5)j} -2\, D_{(19)}\, h_{(6)j}$$

$$A_{(15)j} = D_{(15)/j} + (D_{(4)} – 2\, D_{(10)})\, h_{(5)j} + D_{(14)}\, h_{(1)j} – D_{(16)}\, h_{(2)j} -2\, D_{(18)}\, h_{(4)j} – 2\, D_{(20)}\, h_{(6)j}$$

$$A_{(16)j} = D_{(16)/j} + (D_{(4)} – 2\, D_{(13)})h_{(6)j} – D_{(14)}\, h_{(3)j} + D_{(15)}\, h_{(2)j} – 2\, D_{(19)}\, h_{(4)j} – 2\, D_{(20)}\, h_{(5)j}$$

$$A_{(17)j} = D_{(17)/j} – D_{(5)}\, h_{(3)j} + (D_{(8)} – D_{(11)})h_{(2)j} + (D_{(6)} – D_{(9)})h_{(1)j} + D_{(12)}\, h_{(3)j} + D_{(18)}\, h_{(6)j}$$

$+ D_{(19)} h_{(5)j} + D_{(20)} h_{(4)j}$

$A_{(18)j} = D_{(18)/j} – (D_{(5)} – D_{(15)})h_{(4)j} – (D_{(8)} – D_{(14)}) h_{(5)j} – D_{(17)} h_{(6)j} + (D_{(7)} – D_{(10)})h_{(1)j}$

$– D_{(19)} h_{(2)j} + D_{(20)} h_{(3)j}$

$A_{(19)j} = D_{(19)/j} – D_{(17)} h_{(5)j} - (D_{(7)} – D_{(13)})h_{(3)j} – (D_{(6)} – D_{(16)})h_{(4)j} – (D_{(11)} – D_{(14)})h_{(6)j}$

$+ D_{(18)} h_{(2)j} – D_{(20)} h_{(1)j}$

$A_{(20)j} = D_{(20)/j} + (D_{(10)} – D_{(13)}) h_{(2)j} – (D_{(9)} – D_{(16)})h_{(5)j} – D_{(17)} h_{(4)j} - (D_{(12)} – D_{(15)}) h_{(6)j}$

$– D_{(18)} h_{(3)j} + D_{(19)} h_{(1)j}$ (68)

From equation (4.1), we can obtain by virtue of ${}^1D_{ijk/h}\, l^h = {}^1D_{ijk/0} = {}^1Q_{ijk}$

${}^1Q_{ijk} = A_{(1)0}\, m_i\, m_j\, m_k + A_{(2)0}\, n_{(1)i}\, n_{(1)j}\, n_{(1)k} + A_{(3)0}\, n_{(2)i} n_{(2)j}\, n_{(2)k} + A_{(4)0}\, n_{(3)I}\, n_{(3)j}\, n_{(3)k}$

$+ \sum_{(I,j,k)} [A_{(5)0} \{ m_i\, m_j\, n_{(1)k}\} + A_{(6)0} \{ m_i\, m_j\, n_{(2)k}\} + A_{(7)0} \{ m_i\, m_j\, n_{(3)k}\}$

$+ A_{(8)0} \{ n_{(1)i}\, n_{(1)jj}\, m_k\} + A_{(9)0} \{ n_{(1)i}\, n_{(1)j}\, n_{(2)k}\} + A_{(10)0} \{n_{(1)i}\, n_{(1)j}\, n_{(3)k}\}$

$+ A_{(11)0} \{n_{(2)i}\, n_{(2)j}\, m_k\}+ A_{(12)0} \{n_{(2)i}\, n_{(2)j}\, n_{(1)k}\} + A_{(13)0} \{n_{(2)i}\, n_{(2)j}\, n_{(3)k}\}$

$+ A_{(14)0} \{n_{(3)i}\, n_{(3)j}\, m_k\} + A_{(15)0} \{n_{(3)i}\, n_{(3)j}\, n_{(1)k}\}+ A_{(16)0} \{n_{(3)i}\, n_{(3)j}\, n_{(2)k}\}$

$+ A_{(17)0} \{m_i(n_{(1)j}\, n_{(2)k} + n_{(1)k}\, n_{(2)j})\} + A_{(18)0}\{m_i(n_{(1j}\, n_{(3)k} + n_{(1)k}\, n_{(3)j})\}$

$+ A_{(19)0} \{m_i(n_{(2)j}\, n_{(3)k} + n_{(2)k}\, n_{(3)j})\} + A_{(20)0} \{n_{(1)i}(n_{(2)j}\, n_{(3)k} + n_{(2)k}\, n_{(3)j})\}]$ (69)

Def. 4.1.: If $X^i(x)$ is a vector field satisfying $X^i_{/j} = - \delta^i_j$, it shall be called Q-concurrent vector field of first kind in a five-dimensional Finsler space F^5, if for a scalar μ, it satisfies

$X^i\, {}^1Q_{ijk} = \mu\, h_{jk}$ (70)

From equation (28), we can easily obtain equation (70), which shows:

Theorem 4.1.: If $X^i(x)$ is a D-concurrent vector field of first kind in a five-dimensional Finsler space F^5, it is also Q-concurrent vector field of first kind, such that scalar μ satisfies $\mu = \lambda_{/0}$, but the converse is not true in general.

Equation (70) can alternatively be expressed as

$\mu\, h_{jk} = m_j\, m_k\, \{\beta\, A_{(1)0} + \gamma\, A_{(5)0} + \Theta\, A_{(6)0} + \varphi\, A_{(7)0}\} + n_{(1)j}\, n_{(1)k}\, \{\beta\, A_{(8)0} + \gamma\, A_{(2)0} + \Theta\, A_{(9)0} + \varphi\, A_{(10)0}\}$

$+ n_{(2)j}\, n_{(2)k}\, \{\beta\, A_{(11)0} + \gamma\, A_{(12)0} + \Theta\, A_{(3)0} + \varphi\, A_{(13)0}\} + n_{(3)j}\, n_{(3)k}\{\beta\, A_{(14)0} + \gamma\, A_{(15)0} + \Theta\, A_{(16)0} + \varphi\, A_{(4)0}\}$

$+ (m_j\, n_{(1)k} + m_k\, n_{(1)j})\, \{\beta\, A_{(5)0} + \gamma\, A_{(8)0} + \Theta\, A_{(17)0} + \varphi\, A_{(18)0}\}$

$+ (m_j\, n_{(2)k} + m_k\, n_{(2)j})\, \{\beta\, A_{(6)0} + \gamma\, A_{(17)0} + \Theta\, A_{(11)0} + \varphi\, A_{(19)0}\}$

$+ (m_j\, n_{(3)k} + m_k\, n_{(3)j})\, \{\beta\, A_{(7)0} + \gamma\, A_{(18)0} + \Theta\, A_{(19)0} + \varphi\, A_{(14)0}\}$

$+ (n_{(1)j}\, n_{(2)k} + n_{(1)k}\, n_{(2)j})\, \{\beta\, A_{(17)0} + \gamma\, A_{(9)0} + \Theta\, A_{(12)0} + \varphi\, A_{(20)0}\}$

$+ (n_{(2)j}\, n_{(3)k} + n_{(2)k}\, n_{(3)j})\, \{\beta\, A_{(19)0} + \gamma\, A_{(20)0} + \Theta\, A_{(13)0} + \varphi\, A_{(16)0}\}$

$$+ (n_{(3)j}\, n_{(1)k} + n_{(3)k}\, n_{(1)j})\, \{\beta\, A_{(18)0} + \gamma\, A_{(10)0} + \Theta\, A_{(20)0} + \varphi\, A_{(15)0}\} \quad (71)$$

Multiplying equation (4.5) by m^j, $n_{(1)}{}^j$, $n_{(2)}{}^j$ and $n_{(3)}{}^j$ respectively, we get

$$\mu\, m_k = m_k\, \{\beta\, A_{(1)0} + \gamma\, A_{(5)0} + \Theta\, A_{(6)0} + \varphi\, A_{(7)0}\} + n_{(1)k}\, \{\beta\, A_{(5)0} + \gamma\, A_{(8)0} + \Theta\, A_{(17)0} + \varphi\, A_{(18)0}\}$$

$$+ n_{(2)k}\, \{\beta\, A_{(6)0} + \gamma\, A_{(17)0} + \Theta\, A_{(11)0} + \varphi\, A_{(19)0}\} + n_{(3)k}\{\beta\, A_{(7)0} + \gamma\, A_{(18)0} + \Theta\, A_{(19)0} + \varphi\, A_{(14)0}\}, \quad (72)$$

$$\mu\, n_{(1)k} = m_k\, \{\beta\, A_{(5)0} + \gamma\, A_{(8)0} + \Theta\, A_{(17)0} + \varphi\, A_{(18)0}\} + n_{(1)k}\, \{\beta\, A_{(8)0} + \gamma\, A_{(2)0} + \Theta\, A_{(9)0} + \varphi\, A_{(10)0}\}$$

$$+ n_{(2)k}\{\beta\, A_{(17)0} + \gamma\, A_{(9)0} + \Theta\, A_{(12)0} + \varphi\, A_{(20)0}\} + n_{(3)k}\{\beta\, A_{(18)0} + \gamma\, A_{(10)0} + \Theta\, A_{(20)0} + \varphi\, A_{(15)0}\}, \quad (73)$$

$$\mu\, n_{(2)k} = m_k\, \{\beta\, A_{(6)0} + \gamma\, A_{(17)0} + \Theta\, A_{(11)0} + \varphi\, A_{(19)0}\} + n_{(1)k}\{\beta\, A_{(17)0} + \gamma\, A_{(9)0} + \Theta\, A_{(12)0} + \varphi\, A_{(20)0}\}$$

$$+ n_{(2)k}\{\beta\, A_{(11)0} + \gamma\, A_{(12)0} + \Theta\, A_{(3)0} + \varphi\, A_{(13)0}\} + n_{(3)k}\{\beta\, A_{(19)0} + \gamma\, A_{(20)0} + \Theta\, A_{(13)0} + \varphi\, A_{(16)0}\}, \quad (74)$$

$$\mu\, n_{(3)k} = m_k\, \{\beta\, A_{(7)0} + \gamma\, A_{(18)0} + \Theta\, A_{(19)0} + \varphi\, A_{(14)0}\} + n_{(1)k}\, \{\beta\, A_{(18)0} + \gamma\, A_{(10)0} + \Theta\, A_{(20)0} + \varphi\, A_{(15)0}\}$$

$$+ n_{(2)k}\, \{\beta\, A_{(19)0} + \gamma\, A_{(20)0} + \Theta\, A_{(13)0} + \varphi\, A_{(16)0}\} + n_{(3)k}\, \{\beta\, A_{(14)0} + \gamma\, A_{(15)0} + \Theta\, A_{(16)0} + \varphi\, A_{(4)0}\}. \quad (75)$$

Equations (72) (73) (74) (75) lead to

$$\beta\, A_{(5)0} + \gamma\, A_{(8)0} + \Theta\, A_{(17)0} + \varphi\, A_{(18)0} = \beta\, A_{(6)0} + \gamma\, A_{(17)0} + \Theta\, A_{(11)0} + \varphi\, A_{(19)0}$$

$$= \beta\, A_{(7)0} + \gamma\, A_{(18)0} + \Theta\, A_{(19)0} + \varphi\, A_{(14)0} = \beta\, A_{(17)0} + \gamma\, A_{(9)0} + \Theta\, A_{(12)0} + \varphi\, A_{(20)0}$$

$$= \beta\, A_{(18)0} + \gamma\, A_{(10)0} + \Theta\, A_{(20)0} + \varphi\, A_{(15)0} = \beta\, A_{(19)0} + \gamma\, A_{(20)0} + \Theta\, A_{(13)0} + \varphi\, A_{(16)0} = 0. \quad (76)$$

Eliminating β, γ, Θ, and φ from equation (76), we can obtain following determinant

$$\begin{vmatrix} A_{(5)0} & A_{(8)0} & A_{(17)0} & A_{(18)0} \\ A_{(6)0} & A_{(17)0} & A_{(11)0} & A_{(19)0} \\ A_{(7)0} & A_{(18)0} & A_{(19)0} & A_{(14)0} \\ A_{(17)0} & A_{(9)0} & A_{(12)0} & A_{(20)0} \end{vmatrix} = 0 \quad (77)$$

Hence:

Theorem 4.2.: If $X^i(x)$ is a Q-concurrent vector field of first kind in a five-dimensional Finsler space F^5, its coefficients satisfy determinant (77).

From equations (72) (73) (74) (75), we can also obtain

$$\mu = \beta\, A_{(1)0} + \gamma\, A_{(5)0} + \Theta\, A_{(6)0} + \varphi\, A_{(7)0} = \beta\, A_{(8)0} + \gamma\, A_{(2)0} + \Theta\, A_{(9)0} + \varphi\, A_{(10)0}$$

$$= \beta\, A_{(11)0} + \gamma\, A_{(12)0} + \Theta\, A_{(3)0} + \varphi\, A_{(13)0} = \beta\, A_{(14)0} + \gamma\, A_{(15)0} + \Theta\, A_{(16)0} + \varphi\, A_{(4)0} \quad (78)$$

Multiplying equation (69) by g^{jk}, we get

$${}^1Q_i = m_i(A_{(1)0} + A_{(8)/0} + A_{(11)0} + A_{(14)0}) + n_{(1)i}(A_{(2)0} + A_{(5)0} + A_{(12)0} + A_{(15)0})$$

$$+ n_{(2)i}(A_{(3)0} + A_{(6)0} + A_{(9)0} + A_{(16)0}) + n_{(3)I}(A_{(4)0} + A_{(7)0} + A_{(10)0} + A_{(13)0}) \quad (79)$$

It is known that ${}^1Q_i = {}^1D_{i/0} = ({}^1D\, n_{(1)})_{/0}$, which by virtue of equation (6) can be expressed as

$$^1Q_i = {}^1D_{/0}\, n_{(1)I} + {}^1D(- m_i\, h_0 + n_{(2)I}\, k_0 - n_{(3)I}\, s_0) \quad (80)$$

Comparing equations (79) and (80), we can obtain

$$A_{(1)0} + A_{(8)/0} + A_{(11)0} + A_{(14)0} = - {}^1D\, h_0,\ A_{(2)0} + A_{(5)0} + A_{(12)0} + A_{(15)0} = {}^1D_{/0},$$

$$A_{(3)0} + A_{(6)0} + A_{(9)0} + A_{(16)0} = {}^1D\, k_0,\ A_{(4)0} + A_{(7)0} + A_{(10)0} + A_{(13)0} = - {}^1D\, s_0 \quad (81)$$

Hence:

Theorem 4.3.: If $X^i(x)$ is a Q-concurrent vector field of first kind in a five-dimensional Finsler space F^5, its coefficients satisfy equation (81).

From equation (4.9), we can also obtain

$$4\,\mu = \beta\,(A_{(1)0} + A_{(8)/0} + A_{(11)0} + A_{(14)0}) + \gamma\,(A_{(2)0} + A_{(5)0} + A_{(12)0} + A_{(15)0})$$

$$+ \Theta\,(A_{(3)0} + A_{(6)0} + A_{(9)0} + A_{(16)0}) + \varphi\,(A_{(4)0} + A_{(7)0} + A_{(10)0} + A_{(13)0}), \quad (82)$$

This, by virtue of (4.12) can be expressed as

$$4\,\mu = \gamma\, {}^1D_{/0} + {}^1D\,(-\beta h_0 + \Theta\, k_0 - \varphi\, s_0) \quad (83)$$

Remark: Equation (81) can easily be obtained from equation (83).

TENSOR $^1D_{ijk//r}$ IN F^5.

Taking V-covariant derivative of equation (25) and using equations (15) and (20), we can obtain

$$X^i_{//r} = l^i\, J_{(1)r} + m^i\, J_{(2)r} + n_{(1)i}{}^i\, J_{(3)r} + n_{(2)}{}^I\, J_{(4)r} + n_{(5)}{}^I\, J_{(5)r} \quad (84)$$

Where,

$$J_{(1)r} = \alpha_{//r} - L^{-1}(\beta\, m_r + \gamma\, n_{(1)r} + \Theta\, n_{(2)r} + \varphi\, n_{(3)r}),\ J_{(2)r} = \beta_{//r} - L^{-1}(\gamma\, Q_r + \Theta\, R_r + \varphi\, S_r - \alpha\, m_r),$$

$$J_{(3)r} = \gamma_{//r} + L^{-1}(\beta\, Q_r - \Theta\, U_r - \varphi\, V_r + \alpha\, n_{(1)r}),\ J_{(4)r} = \Theta_{//r} + L^{-1}(\beta\, R_r + \gamma\, U_r - \varphi\, X_r + \alpha\, n_{(2)r}),$$

$$J_{(5)r} = \varphi_{//r} + L^{-1}(\beta\, S_r + \gamma\, V_r + \Theta\, X_r + \alpha\, n_{(3)r}). \quad (85)$$

From these equations we can obtain by virtue of equation (32) following relations:

α//r = L-1(β mr + γ n(1)r + Θ n(2)r + φ n(3)r – α lr) (86)

$$\beta_{//r} = L^{-1}\{m_r\,(\beta\, C_{(1)} + \gamma\, C_{(5)} + \Theta\, C_{(6)} + \varphi\, C_{(7)} - \alpha) + n_{(1)r}(\beta\, C_{(5)} + \gamma\, C_{(8)} + \Theta\, C_{(17)} + \varphi\, C_{(18)})$$

$$+ n_{(2)r}(\beta\, C_{(6)} + \gamma\, C_{(17)} + \Theta\, C_{(11)} + \varphi\, C_{(19)}) + n_{(3)r}(\beta\, C_{(7)} + \gamma\, C_{(18)} + \Theta\, C_{(19)} + \varphi\, C_{(4)})$$

$$+ \gamma\, Q_r + \Theta\, R_r + \varphi\, S_r\} \quad (87)$$

$$\gamma_{//r} = L^{-1}\{m_r(\beta C_{(5)} + \gamma\, C_{(8)} + \Theta\, C_{(17)} + \varphi\, C_{(18)}) + n_{(1)r}(\beta\, C_{(8)} + \gamma\, C_{(2)} + \Theta\, C_{(9)} + \varphi\, C_{(10)} - \alpha)$$

$$+ n_{(2)r}(\beta\, C_{(17)} + \gamma\, C_{(9)} + \Theta\, C_{(12)} + \varphi\, C_{(20)}) + n_{(3)r}(\beta\, C_{(18)} + \gamma\, C_{(10)} + \Theta\, C_{(20)} + \varphi\, C_{(15)})$$

$$+ \varphi\, V_r - \beta\, Q_r - \Theta\, U_r\} \quad (88)$$

$$\Theta_{//r} = L^{-1}\{m_r(\beta\, C_{(6)} + \gamma\, C_{(17)} + \Theta\, C_{(11)} + \varphi\, C_{(19)}) + n_{(1)r}(\beta\, C_{(17)} + \gamma\, C_{(9)} + \Theta\, C_{(12)} + \varphi\, C_{(20)})$$

$+ n_{(2)r}(\beta C_{(11)} + \gamma C_{(12)} + \Theta C_{(3)} + \varphi C_{(13)} - \alpha) + n_{(3)r}(\beta C_{(19)} + \gamma C_{(20)} + \Theta C_{(13)} + \varphi C_{(16)})$

$+ \varphi X_r - \beta R_r - \gamma U_r\}$ (89)

$\varphi_{//r} = L^{-1}\{m_r(\beta C_{(7)} + \gamma C_{(18)} + \Theta C_{(19)} + \varphi C_{(14)}) + n_{(1)r}(\beta C_{(18)} + \gamma C_{(10)} + \Theta C_{(20)} + \varphi C_{(15)})$

$+ n_{(2)r}(\beta C_{(19)} + \gamma C_{(20)} + \Theta C_{(13)} + \varphi C_{(16)}) + n_{(3)r}(\beta C_{(14)} + \gamma C_{(15)} + \Theta C_{(16)} + \varphi C_{(4)} - \alpha)$

$-\beta S_r - \gamma V_r - \Theta X_r\}$ (90)

From equations (86) (87) (88) (89) (90) with the help of equations (8) and (9) (10) (11) (12) (13) (14) we can obtain

$\alpha_{//r} l^r = - L^{-1} \alpha,\ \beta_{//r} l^r = 0,\ \gamma_{//r} l^r = 0,\ \Theta_{//r} l^r = 0,\ \varphi_{//r} l^r = 0,$ (91)

$\alpha_{//r} m^r = L^{-1} \beta,\ \beta_{//r} m^r = L^{-1}(\beta C_{(1)} + \Theta C_{(6)} + \varphi C_{(7)} - \alpha + \gamma V_{2)32} + \Theta V_{2)42} + \varphi V_{2)52}),$

$\gamma_{//r} m^r = L^{-1}(\beta C_{(5)} + \gamma C_{(8)} + \Theta C_{(17)} + \varphi C_{(18)} - \beta V_{2)32} - \Theta V_{3)42} + \varphi V_{3)52}),$

$\Theta_{//r} m^r = L^{-1}(\beta C_{(6)} + \gamma C_{(17)} + \Theta C_{(11)} + \varphi C_{(19)} - \beta V_{2)42} - \gamma V_{3)42} + \varphi V_{4)52}),$

$\varphi_{//r} m^r = L^{-1}(\beta C_{(7)} + \gamma C_{(18)} + \Theta C_{(19)} + \varphi C_{(14)} - \beta V_{2)52} - \gamma V_{3)52} - \Theta V_{4)52}),$ (92)

$\alpha_{//r} n_{(1)}{}^r = L^{-1} \gamma,\ \beta_{//r} n_{(1)}{}^r = L^{-1} (\beta C_{(5)} + \gamma C_{(8)} + \Theta C_{(17)} + \varphi C_{(18)} + \gamma V_{2)33} + \Theta V_{2)43} + \varphi V_{2)53}),$

$\gamma_{//r} n_{(1)}{}^r = L^{-1}(\beta C_{(8)} + \gamma C_{(2)} + \Theta C_{(9)} + \varphi C_{(10)} - \alpha - \beta V_{2)33} - \Theta V_{3)43} + \varphi V_{3)53}),$

$\Theta_{//r} n_{(1)}{}^r = L^{-1}(\beta C_{(17)} + \gamma C_{(9)} + \Theta C_{(12)} + \varphi C_{(20)} - \beta V_{2)43} - \gamma V_{3)43} + \varphi V_{4)53}),$

$\varphi_{//r} n_{(1)}{}^r = L^{-1}(\beta C_{(18)} + \gamma C_{(10)} + \Theta C_{(20)} + \varphi C_{(15)} - \beta V_{2)53} - \gamma V_{3)53} - \Theta V_{4)53}),$ (93)

$\alpha_{//r} n_{(2)}{}^r = L^{-1}\Theta,\ \beta_{//r} n_{(2)}{}^r = L^{-1}(\beta C_{(6)} + \gamma C_{(17)} + \Theta C_{(11)} + \varphi C_{(19)} + \gamma V_{2)34} + \Theta V_{2)44} + \varphi V_{2)54}),$

$\gamma_{//r} n_{(2)}{}^r = L^{-1}(\beta C_{(17)} + \gamma C_{(9)} + \Theta C_{(12)} + \varphi C_{(20)} - \beta V_{2)34} - \Theta V_{3)44} + \varphi V_{3)54}),$

$\Theta_{//r} n_{(2)}{}^r = L^{-1}(\beta C_{(11)} + \gamma C_{(12)} + \Theta C_{(3)} + \varphi C_{(13)} - \alpha - \beta V_{2)44} - \gamma V_{3)44} + \varphi V_{4)54}),$

$\varphi_{//r} n_{(2)}{}^r = L^{-1}(\beta C_{(19)} + \gamma C_{(20)} + \Theta C_{(13)} + \varphi C_{(16)} - \beta V_{2)54} - \gamma V_{3)54} - \Theta V_{4)54}),$ (94)

$\alpha_{//r} n_{(3)}{}^r = L^{-1}\varphi,\ \beta_{//r} n_{(3)}{}^r = L^{-1}(\beta C_{(7)} + \gamma C_{(18)} + \Theta C_{(19)} + \varphi C_{(4)} - \beta V_{2)35} - \Theta V_{3)45} + \varphi V_{3)55}),$

$\gamma_{//r} n_{(3)}{}^r = L^{-1}(\beta C_{(18)} + \gamma C_{(10)} + \Theta C_{(20)} + \varphi C_{(15)} - \beta V_{2)35} - \Theta V_{3)45} + \varphi V_{3)55}),$

$\Theta_{//r} n_{(3)}{}^r = L^{--1} (\beta C_{(19)} + \gamma C_{(20)} + \Theta C_{(13)} + \varphi C_{(16)} - \beta V_{2)45} - \gamma V_{3)45} + \varphi V_{4)55}),$

$\varphi_{//r} n_{(3)}{}^r = L^{-1}(\beta C_{(14)} + \gamma C_{(15)} + \Theta C_{(16)} + \varphi C_{(4)} - \alpha - \beta V_{2)55} - \gamma V_{3)55} - \Theta V_{4)55}).$ (95)

Hence:

Theorem 5.1.: In a five-dimensional Finsler space F^5, for a vector field X^i, given by equation (25), its coefficients satisfy equations (5.4) a, b, c, d, e.

Taking V-covariant derivative of equation (22) and using equations (15) (16) (17) (18) (19), we get

$^1D_{ijk//r} = \sum_{(I,j,k)} \{m_j m_k\, {}^1T_{ir} + n_{(1)j} n_{(1)k}\, {}^2T_{ir} + n_{(2)j} n_{(2)k}\, {}^3T_{ir} + n_{(3)j} n_{(3)k}\, {}^4T_{ir} + (m_j n_{(1)k} + m_k n_{(1)j})\, {}^5T_{ir}$

$+ (m_j n_{(2)k} + m_k n_{(2)j})\, {}^6T_{ir} + (m_j n_{(3)k} + m_k n_{(3)j})\, {}^7T_{ir} + (n_{(1)j} n_{(2)k} + n_{(1)k} n_{(2)j})\, {}^8T_{ir} +$

$+ (n_{(1)j}\, n_{(3)k} + n_{(1)k}\, n_{(3)j})\, {}^{9}T_{ir} + (n_{(2)j}\, n_{(3)k} + n_{(2)k}\, n_{(3)j})\, {}^{10}T_{ir}\}$ (96)

Where,

${}^{1}T_{ir} = \{(1/3)\, D_{(1)//r} - L^{-1}(D_{(5)}\, Q_r + D_{(6)}\, R_r + D_{(7)}\, S_r)\}\, m_i + L^{-1}\{n_{(1)i}\, (D_{(1)}\, Q_r - D_{(6)}\, U_r - D_{(7)}\, V_r)$

$+ n_{(2)i}(D_{(1)}\, R_r + D_{(5)}\, U_r - D_{(7)}\, X_r) + n_{(3)i}(D_{(1)}\, S_r + D_{(5)}\, V_r + D_{(6)}\, X_r) - l_i(D_{(1)}\, m_r + D_{(5)}\, n_{(1)r}$

$+ D_{(6)}\, n_{(2)r} + D_{(7)}\, n_{(3)r})\}$, (97)

${}^{2}T_{ir} = \{(1/3)\, D_{(2)//r} + L^{-1}(D_{(8)}\, Q_r - D_{(9)}\, U_r - D_{(10)}\, V_r)\}\, n_{(1)I} - L^{-1}\{m_i(D_{(2)}\, Q_r + D_{(9)}\, R_r + D_{(10)}\, S_r)$

$- n_{(2)i}(D_{(2)}\, U_r + D_{(8)}\, R_r - D_{(10)}\, X_r) - n_{(3)i}(D_{(2)}\, V_r + D_{(8)}\, S_r + D_{(9)}\, X_r) + l_i(D_{(2)}\, n_{(1)r} + D_{(8)}\, m_r$

$+ D_{(9)}\, n_{(2)r} + D_{(10)}\, n_{(3)r})\}$, (98)

${}^{3}T_{ir} = \{(1/3)\, D_{(3)//r} + L^{-1}(D_{(11)}\, R_r + D_{(12)}\, U_r - D_{(13)}\, X_r)\}\, n_{(2)I} - L^{-1}\{m_i(D_{(3)}\, R_r + D_{(12)}\, Q_r + D_{(13)}\, S_r)$

$+ n_{(1)i}(D_{(3)}\, U_r - D_{(11)}\, Q_r + D_{(13)}\, V_r) - n_{(3)i}(D_{(3)}\, X_r + D_{(11)}\, S_r + D_{(12)}\, V_r) + l_i(D_{(11)}\, m_r + D_{(12)}\, n_{(1)r}$

$+ D_{(3)}\, n_{(2)r} + D_{(13)}\, n_{(3)r})\}$, (99)

${}^{4}T_{ir} = \{(1/3)\, D_{(4)//r} + L^{-1}(D_{(14)}\, S_r + D_{(15)}\, V_r + D_{(16)}\, X_r)\}\, n_{(3)I} - L^{-1}\{m_i(D_{(4)}\, S_r + D_{(15)}\, Q_r + D_{(16)}\, R_r)$

$+ n_{(1)I}\, (D_{(4)}\, V_r - D_{(14)}\, Q_r + D_{(16)}\, U_r) + n_{(2)I}\, (D_{(4)}\, X_r - D_{(14)}\, R_r - D_{(15)}\, U_r) + l_i(D_{(14)}\, m_r + D_{(15)}\, n_{(1)r}$

$+ D_{(16)}\, n_{(2)r} + D_{(4)}\, n_{(3)r})\}$, (100)

${}^{5}T_{ir} = \{(1/3)\, D_{(5)//r} - L^{-1}(D_{(8)}\, Q_r + D_{(17)}\, R_r + D_{(19)}\, S_r)\}\, m_i + \{(1/3)\, D_{(8)//r} + L^{-1}(D_{(5)}\, Q_r - D_{(17)}\, U_r$

$-D_{(19)}\, V_r)\}\, n_{(1)I} + \{(1/3)\, D_{(17)//r} + L^{-1}(D_{(5)}\, R_r + D_{(8)}\, U_r - D_{(19)}\, X_r)\}\, n_{(2)I} + \{(1/3)\, D_{(19)//r}$

$+ L^{-1}(D_{(5)}\, S_r + D_{(8)}\, V_r + D_{(17)}\, X_r)\}\, n_{(3)I} - L^{-1}(D_{(5)}\, m_r + D_{(8)}\, n_{(1)r} + D_{(17)}\, n_{(2)r} + D_{(19)}\, n_{(3)r})\quad l_i$ (101)

${}^{6}T_{ir} = \{(1/3)\, D_{(6)//r} - L^{-1}(D_{(11)}\, R_r + D_{(17)}\, Q_r + D_{(18)}\, S_r)\}\, m_i + \{(1/3)\, D_{(17)//r} + L^{-1}(D_{(6)}\, Q_r - D_{(11)}\, U_r$

$-D_{(18)}\, V_r)\}\, n_{(1)I} + \{(1/3)\, D_{(11)//r} + L^{-1}(D_{(6)}\, R_r + D_{(17)}\, U_r - D_{(18)}\, V_r)\}\, n_{(2)I} + \{(1/3)\, D_{(18)//r}$

$+ L^{-1}(D_{(6)}\, S_r + D_{(11)}\, X_r + D_{(17)}\, V_r)\}\, n_{(3)I} - L^{-1}(D_{(6)}\, m_r + D_{(17)}\, n_{(1)r} + D_{(11)}\, n_{(2)r} + D_{(18)}\, n_{(3)r})\, l_i$ (102)

${}^{7}T_{ir} = \{(1/3)\, D_{(7)//r} - L^{-1}(D_{(14)}\, S_r + D_{(18)}\, R_r + D_{(19)}\, Q_r)\}\, m_i + \{(1/3)\, D_{(19)//r} + L^{-1}(D_{(7)}\, Q_r - D_{(14)}\, V_r$

$-D_{(18)}\, U_r)\}\, n_{(1)I} + \{(1/3)\, D_{(18)//r} + L^{-1}(D_{(7)}\, R_r - D_{(14)}\, X_r + D_{(19)}\, U_r)\}\, n_{(2)I} + \{(1/3)\, D_{(14)//r}$

$+ L^{-1}(D_{(7)}\, S_r + D_{(18)}\, X_r + D_{(19)}\, V_r)\}\, n_{(3)I} - L^{-1}(D_{(7)}\, m_r + D_{(19)}\, n_{(1)r} + D_{(18)}\, n_{(2)r} + D_{(14)}\, n_{(3)r})\, l_i$ (103)

${}^{8}T_{ir} = \{(1/3)\, D_{(17)//r} - L^{-1}(D_{(9)}\, Q_r + D_{(12)}\, R_r + D_{(20)}\, S_r)\}\, m_i + \{(1/3)\, D_{(9)//r} - L^{-1}(D_{(12)}\, U_r - D_{(17)}\, Q_r$

$+ D_{(20)}\, V_r)\}\, n_{(1)I} + \{(1/3)\, D_{(12)//r} + L^{-1}(D_{(9)}\, U_r + D_{(17)}\, R_r - D_{(20)}\, X_r)\}\, n_{(2)I} + \{(1/3)\, D_{(20)//r}$

$+ L^{-1}(D_{(9)}\, V_r + D_{(12)}\, X_r + D_{(17)}\, S_r)\}\, n_{(3)I} - L^{-1}(D_{(17)}\, m_r + D_{(9)}\, n_{(1)r} + D_{(12)}\, n_{(2)r} + D_{(20)}\, n_{(3)r})\, l_i$ (104)

${}^{9}T_{ir} = \{(1/3)\, D_{(19)//r} - L^{-1}(D_{(10)}\, Q_r + D_{(15)}\, S_r + D_{(20)}\, R_r)\}\, m_i + \{(1/3)\, D_{(10)//r} - L^{-1}(D_{(15)}\, V_r - D_{(19)}\, Q_r$

$+ D_{(20)}\, U_r)\, n_{(1)I} + \{(1/3)\, D_{(20)//r} + L^{-1}(D_{(10)}\, U_r - D_{(15)}\, X_r + D_{(19)}\, R_r)\}\, n_{(2)I} + \{(1/3)\, D_{(15)//r}$

$+ L^{-1}(D_{(10)}\, V_r + D_{(19)}\, S_r + D_{(20)}\, X_r)\}\, n_{(3)I} - L^{-1}(D_{(10)}\, n_{(1)r} + D_{(15)}\, n_{(3)r} + D_{(19)}\, m_r + D_{(20)}\, n_{(2)r})\, l_i$ (105)

$$^{10}T_{ir} = \{(1/3)\, D_{(18)//r} - L^{-1}(D_{(13)}\, R_r + D_{(16)}\, S_r + D_{(20)}\, Q_r)\}\, m_i + \{(1/3)\, D_{(20)//r} - L^{-1}(D_{(13)}\, U_r + D_{(16)}\, V_r$$

$$- D_{(18)}\, Q_r)\}\, n_{(1)I} + \{(1/3)\, D_{(13)//r} - L^{-1}(D_{(16)}\, X_r - D_{(18)}\, R_r - D_{(20)}\, U_r)\}\, n_{(20i} + \{(1/3)\, D_{(16)//r}$$

$$+ L^{-1}(D_{(13)}\, X_r + D_{(18)}\, S_r + D_{(20)}\, V_r)\}\, n_{(3)I} - L^{-1}(D_{(13)}\, n_{(2)r} + D_{(16)}\, n_{(3)r} + D_{(18)}\, m_r + D_{(20)}\, n_{(1)r})\, l_i \quad (106)$$

Hence:

Theorem 5.2.: In a five-dimensional Finsler space F^5, v- covariant derivative of the tensor $^1D_{ijk}$ given by the equation (22), is expressed as in (96), where tensors $^1T_{ir}$, $^2T_{ir}$,…, $^{10}T_{ir}$ are given by equations (97), (98),…, (106) respectively.

Tensor $^1D_{ijkh}$ IN F^5

We here define a tensor $^1D_{ijkh}$ as follows:

$$^1D_{ijkh} = Ç_{(h,,k)}\{^1D_{ihr}\; ^1D^r{}_{jk}\} \quad (107)$$

Substituting the value of $^1D_{ijk}$ in equation (107), we can obtain on simplification

$$^1D_{ijkh} = Ç_{(h,k)}\, [\, m_j\, m_k\, \{D_{(1)}\, ^1B_{ih} + D_{(5)}\, ^2B_{ih} + D_{(6)}\, ^3B_{ih} + D_{(7)}\, ^4B_{ih}\}$$

$$+ n_{(1)j}\, n_{(1)k}\{D_{(2)}\, ^2B_{ih} + D_{(8)}\, ^1B_{ih} + D_{(9)}\, ^3B_{ih} + D_{(10)}\, ^4B_{ih}\}$$

$$+ n_{(2)j}\, n_{(2)k}\{D_{(3)}\, ^3B_{ih} + D_{(11)}\, ^1B_{ih} + D_{(12)}\, ^2B_{ih} + D_{(13)}\, ^4B_{ih}\}$$

$$+ n_{(3)j}\, n_{(3)k}\{D_{(4)}\, ^4B_{ih} + D_{(14)}\, ^1B_{ih} + D_{(15)}\, ^2B_{ih} + D_{(16)}\, ^3B_{ih}\}$$

$$+ (m_j\, n_{(1)k} + m_k\, n_{(1)j})\{D_{(5)}\, ^1B_{ih} + D_{(8)}\, ^2B_{ih} + D_{(17)}\, ^3B_{ih} + D_{(19)}\, ^4B_{ih}\}$$

$$+ (m_j\, n_{(2)k} + m_k\, n_{(2)j})\{D_{(6)}\, ^1B_{ih} + D_{(11)}\, ^3B_{ih} + D_{(17)}\, ^2B_{ih} + D_{(18)}\, ^4B_{ih}\}$$

$$+ (m_j\, n_{(3)k} + m_k\, n_{(3)j})\{D_{(7)}\, ^1B_{ih} + D_{(14)}\, ^4B_{ih} + D_{(18)}\, ^3B_{ih} + D_{(19)}\, ^2B_{ih}\}$$

$$+ (n_{(1)j}\, n_{(2)k} + n_{(1)k}\, n_{(2)j})\{D_{(9)}\, ^2B_{ih} + D_{(12)}\, ^3B_{ih} + D_{(17)}\, ^1B_{ih} + D_{(20)}\, ^4B_{ih}\}$$

$$+ (n_{(2)j}\, n_{(3)k} + n_{(2)k}\, n_{(3)j})\{D_{(13)}\, ^3B_{ih} + D_{(16)}\, ^4B_{ih} + D_{(18)}\, ^1B_{ih} + D_{(20)}\, ^1B_{ih}\}$$

$$+ (n_{(3)j}\, n_{(1)k} + n_{(3)k}\, n_{(1)j})\{D_{(10)}\, ^3B_{ih} + D_{(15)}\, ^4B_{ih} + D_{(19)}\, ^1B_{ih} + D_{(20)}\, ^2B_{ih}\} \quad (108)$$

Where,

$$^1B_{ih} = D_{(1)}\, m_i\, m_h + D_{(5)}\, (m_i\, n_{(1)h} + m_h\, n_{(1)i}) + D_{(6)}\, (m_i\, n_{(2)h} + m_h\, n_{(2)i}) + D_{(7)}\, (m_i\, n_{(3)h} + m_h\, n_{(3)i})$$

$$+ D_{(11)}\, n_{(2)I}\, n_{(2)h} + D_{(14)}\, n_{(3)I}\, n_{(3)h} + D_{(17)}\, (n_{(1)i}\, n_{(2)h} + n_{(1)h}\, n_{(2)i}) + D_{(18)}\, (n_{(2)I}\, n_{(3)h} + n_{(2)h}\, n_{(3)i})$$

$$+ D_{(19)}(n_{(1)I}\, n_{(3)h} + n_{(1)h}\, n_{(3)i}), \quad (109)$$

$$^2B_{ih} = D_{(2)}\, n_{(1)I}\, n_{(1)h} + D_{(5)}\, m_i\, m_h + D_{(8)}\, (m_i\, n_{(1)h} + m_h\, n_{(1)i}) + D_{(9)}\, (n_{(1)I}\, n_{(2)h} + n_{(1)h}\, n_{(2)i})$$

$$+ D_{(10)}\, (n_{(1)I}\, n_{(3)h} + n_{(1)h}\, n_{(3)i}) + D_{(12)}\, n_{(2)I}\, n_{(2)h} + D_{(15)}\, n_{(3)I}\, n_{(3)h} + D_{(17)}\, (m_i\, n_{(2)h} + m_h\, n_{(2)i})$$

$$+ D_{(19)}\, (m_i\, n_{(3)h} + m_h\, n_{(3)i}) + D_{(20)}\, (n_{(2)I}\, n_{(3)h} + n_{(2)h}\, n_{(3)i}), \quad (110)$$

$$^3B_{ih} = D_{(3)}\, n_{(2)I}\, n_{(2)h} + D_{(6)}\, m_i\, m_h + D_{(9)}\, n_{(1)I}\, n_{(1)h} + D_{(11)}\, (m_i\, n_{(2)h} + m_h\, n_{(2i}) + D_{(12)}\, (n_{(1)I}\, n_{(2)h} + n_{(1)h}\, n_{(2)i})$$

$$+ D_{(13)}\, (n_{(2)I}\, n_{(3)h} + n_{(2)h}\, n_{(3)i}) + D_{(16)}\, n_{(3)I}\, n_{(3)h} + D_{(17)}\, (m_i\, n_{(1)h} + m_h\, n_{(1)i})$$

$$+ D_{(18)} (m_i n_{(3)h} + m_h n_{(3)i}) + D_{(20)} (n_{(1)I} n_{(3)h} + n_{(1)h} n_{(3)i}), \quad (111)$$

$$^4B_{ih} = D_{(4)} n_{(3)I} n_{(3)h} + D_{(7)} m_i m_h + D_{(10)} n_{(1)I} n_{(1)h} + D_{(13)} n_{(2)I} n_{(2)h} + D_{(14)} (m_i n_{(3)h} + m_h n_{(3)i})$$

$$+ D_{(15)} (n_{(1)I} n_{(3)h} + n_{(1)h} n_{(3)i}) + D_{(16)} (n_{(2)I} n_{(3)h} + n_{(2)h} n_{(3)i}) + D_{(18)} (m_i n_{(2)h} + m_h n_{(2)i})$$

$$+ D_{(19)} (m_i n_{(1)h} + m_h n_{(1)i}) + D_{(20)} (n_{(1)I} n_{(2)h} + n_{(1)h} n_{(2)i}), \quad (112)$$

Are four symmetric tensors in i and h. These tensors with the help of equation (23) give

$$^1B_{ih} m^h + {}^2B_{ih} n_{(1)}{}^h + {}^3B_{ih} n_{(2)}{}^h + {}^4B_{ih} n_{(3)}{}^h = {}^1D_i \quad (113)$$

If $X^i(x)$ is a D-concurrent vector field of first kind, with the help of equations (28) and (107) we can obtain $X^i\ {}^1D_{ijkh} = 0$, which also leads to $X^i\ {}^1D_{ijkh/m} = {}^1D_{mjkh}$. Hence:

Theorem 6.1.: In a five -dimensional Finsler space F^5, a D-concurrent vector field of first kind satisfies $X^i\ {}^1D_{ijkh} = 0$ and $X^i\ {}^1D_{ijkh/m} = {}^1D_{mjkh}$.

Remarks:

- Tensors $^2D_{ijk}$ and $^3D_{ijk}$ also satisfy properties similar to $^1D_{ijk}$.
- Curvature properties related with these tensors may be studied in the subsequent research work.

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